PHYS 704
Homework 10

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1. A flat right rectangular loop carrying a constant currint I_1 is placed near a long straight wire carrying a current I_2. The loop is oriented so that its center is a perpendicular distance d from the wire; the sides of length a are parallel to the wire and the sides of length b make an angle α with the plane containing the wire and the loop's center. The direction of the current I_1 is the same as that of I_2 in the side of the rectangle nearest the wire.

   (a) Show that the interaction magnetic energy

$$W_{12} = \int \boldsymbol{J} \cdot \boldsymbol{A}_2 d^3x = I_1 F_2$$

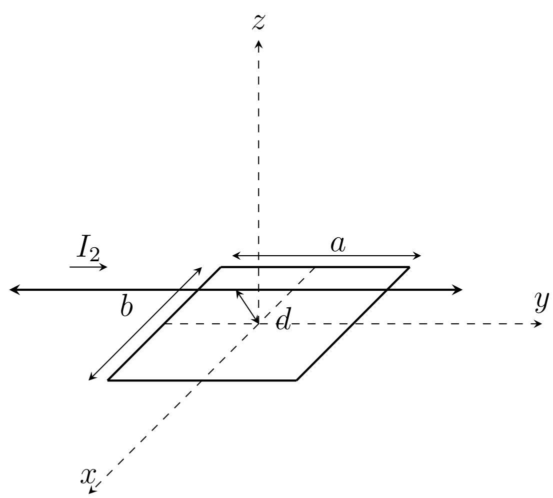
   (where F_2 is the magnetic flux from I_2 linking the rectangular circuit carrying I_1), is

$$W_{12} = \frac{\mu_0 I_1 I_2 a}{4\pi} \ln \left[\frac{4d^2 + b^2 + 4db\cos\alpha}{4d^2 + b^2 - 4db\cos\alpha} \right]$$

   Solution.

$$W_{12} = I_1 \oint d\boldsymbol{l}_1 \cdot \boldsymbol{A}_2$$

   Take the loop to be in the x-y plane, centered at the origin, with the wire above it as shown:



So $\boldsymbol{A}_2$ is given by

$$\boldsymbol{A}_2 = -\frac{I_2\hat{\boldsymbol{y}}}{4\pi}\ln\left[(x - d\cos\alpha)^2(x - d\sin\alpha)^2\right]$$

And only the sides of length a will contribute:

$$\begin{aligned}\Rightarrow W_{12} &= I_1\oint d\boldsymbol{l}_1\cdot\boldsymbol{A}_2\\ &= \frac{\mu_0 I_1 I_2 a}{4\pi}\ln\left[\frac{\left(-\frac{b}{2} - d\cos\alpha\right)^2 - (d\sin\alpha)^2}{\left(\frac{b}{2} - d\cos\alpha\right)^2 + (d\sin\alpha)^2}\right]\\ &= \frac{\mu_0 I_1 I_2 a}{4\pi}\ln\left[\frac{4d^2 + b^2 + 4db\cos\alpha}{4d^2 + b^2 - 4db\cos\alpha}\right]\end{aligned}$$

(b) Calculate the force between the loop and the wire for fixed currents.

Solution.

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$$

Only the force on the sides of length a will be nonzero by symmetry

$$\begin{aligned} B_x(x,0) &= -\left.\frac{\partial A_y}{\partial z}\right|_{z=0} \\ &= \frac{-d\sin\alpha}{(x-d\cos\alpha)^2+(d\sin\alpha)^2} \\ B_z(x,0) &= \left.\frac{\partial A_y}{\partial x}\right|_{z=0} \\ &= \frac{x-d\sin\alpha}{(x-d\cos\alpha)^2+(d\sin\alpha)^2} \end{aligned}$$

$$\begin{aligned} F_x &= I_1\left[B_z\left(\frac{b}{2},0\right)-B_z\left(-\frac{b}{2},0\right)\right] \\ &= \frac{2\mu_0 I_1 I_2 ab}{\pi}\frac{4d^2\cos(2\alpha)-b^2}{b^4-8d^2\cos(2\alpha)b^2+16d^4} \\ F_z &= -I_2\left[B_x\left(\frac{b}{2},0\right)-B_x\left(-\frac{b}{2},0\right)\right] \\ &= \frac{8\mu_0 I_1 I_2 ab}{\pi}\frac{d^2\sin(2\alpha)}{b^4-8d^2\cos(2\alpha)b^2+16d^4} \end{aligned}$$

(c) Repeat the calculation for a circular loop of radius a, whose plane is parallel to the wire and makes an angle α with respect to the plane containing the center of the loop and the wire. Show that the interaction energy is

$$W_{12} = \mu_0 I_1 I_2 d \cdot \Re\left\{e^{i\alpha}-\sqrt{e^{2i\alpha}-a^2/d^2}\right\}$$

Find the force.

Solution. Converting to cylindrical coordinates, $x = a\cos\phi$ and $dl_y = d\phi\, a\cos\phi$:

$$W_{12} = I_1\int d\phi\, a\cos\phi A_y(a\cos\phi,0)$$

Expand A_y in terms of $1/d$:

$$W_{12} = \mu_0 I_1 I_2\left(\frac{a^2\cos\alpha}{2d}+\frac{a^4\cos(3\alpha)}{8d^3}+\cdots\right)$$

$$\frac{1-\sqrt{1-z^2}}{z} = \frac{z}{2} + \frac{z^3}{8} + \cdots$$

therefore

$$\begin{aligned} W_{12} &= \mu_0 a I_1 I_2 \Re\left(\frac{1-\sqrt{1-(\frac{a}{d}e^{i\alpha})^2}}{\frac{a}{d}e^{i\alpha}}\right) \\ &= \mu_0 d I_1 I_2 \Re\left(e^{-i\alpha} - \sqrt{e^{-2i\alpha} - \frac{a^2}{d^2}}\right) \\ &= \mu_0 d I_1 I_2 \Re\left(e^{i\alpha} - \sqrt{e^{2i\alpha} - \frac{a^2}{d^2}}\right) \end{aligned}$$

The last step is obtained from taking the real part.
For the force, a similar process is applied to the following integral:

$$\boldsymbol{F} = \hat{\boldsymbol{x}} I_1 \int d\phi\ a\cos\phi B_z(a\cos\phi, 0) - \hat{\boldsymbol{z}} I_1 \int d\phi\ a\cos\phi B_x(a\cos\phi, 0)$$

yielding

$$\begin{aligned} F_x &= \mu_0 I_1 I_2 \Re\left[\frac{1}{\sqrt{1-(\frac{a}{d}E^{i\alpha})}}\right] \\ F_z &= \mu_0 I_1 I_2 \Im\left[\frac{1}{\sqrt{1-(\frac{a}{d}E^{i\alpha})}}\right] \end{aligned}$$

(d) For both loops, show that when $d \gg a, b$ the interaction energy reduces to $W_{12} \approx \boldsymbol{m} \cdot \boldsymbol{B}$, where $\boldsymbol{m}$ is the magnetic moment of the loop. Explain the sign.

Solution.

$$\begin{aligned} W_{12} &= \frac{\mu_0 I_1 I_2 a}{4\pi} \ln\left[\frac{4d^2+b^2+4db\cos\alpha}{4d^2+b^2-4db\cos\alpha}\right] && (1) \\ &\approx \frac{\mu_0 I_1 I_2 a}{4\pi} \frac{2b\cos(\alpha)}{d} && (2) \\ &= (I_1 ab)\frac{\mu_0 I_2 \cos\alpha}{2\pi d} && (3) \\ &= m_1 B_{2z} && (4) \end{aligned}$$

Similarly for the second case:

$$\begin{aligned} W_{12} &\approx \mu_0 I_1 I_2 a \frac{a\cos\alpha}{2d} \\ &= (I_1 \pi a^2) \frac{\mu_0 I_2 \cos\alpha}{2\pi d} \\ &= m_1 B_{2z} \end{aligned}$$

The sign is positive because the magnetic field and the magnetic moment oppose each other.

2. Show that the mutual inductance of two circular coaxial loops in a homogeneous medium of permeabililty μ is

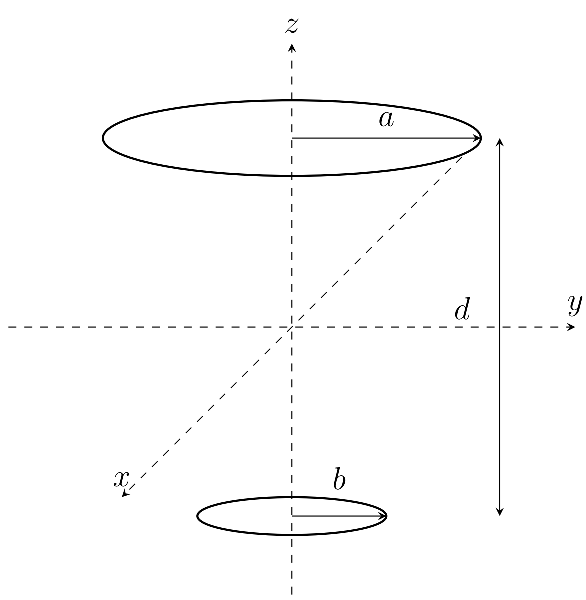
$$M_{12} = \mu\sqrt{ab}\left[\left(\frac{2}{k} - k\right) K(k) - \frac{2}{k} E(k)\right]$$

where

$$k^2 = \frac{4ab}{(a + b^2) + d^2}$$

and a, b are the radii of the loops, d is the distance between their centers, and K and E are the complete elliptic integrals.

Find the limiting value when $d \ll a, b$ and $a \simeq b$



Solution.

$$\begin{aligned}
M_{ij} &= \frac{1}{I_j}\int_{S_i}(\boldsymbol{\nabla}\times\boldsymbol{A}_{ij})\cdot d\boldsymbol{a} \\
&= \frac{1}{I_j}\oint_{C_i}\boldsymbol{A}\cdot d\boldsymbol{s_i} \\
&= \frac{1}{I_j}\oint_{C_i}\left(\frac{\mu I_j}{4\pi}\oint_{C_j}\frac{d\boldsymbol{s}_j}{r}\right)\cdot d\boldsymbol{s_i} \\
&= \frac{\mu}{4\pi}\oint_{C_i}\oint_{C_j} d\boldsymbol{s}_j\cdot d\boldsymbol{s}_i
\end{aligned}$$

$$\begin{aligned}
d\boldsymbol{s}_1 &= a(-\hat{\boldsymbol{x}}\sin\phi_1+\hat{\boldsymbol{y}}\cos\phi_1)d\phi_1 \\
d\boldsymbol{s}_2 &= b(-\hat{\boldsymbol{x}}\sin\phi_2+\hat{\boldsymbol{y}}\cos\phi_2)d\phi_2 \\
\Rightarrow d\boldsymbol{s}_1\cdot d\boldsymbol{s}_2 &= ab\left(\sin\phi_1\sin\phi_2+\cos\phi_1\cos\phi_2\right)d\phi_1 d\phi_2 \\
&= ab\cos(\phi_1-\phi_2)d\phi_1 d\phi_2
\end{aligned}$$

$$\begin{aligned}
r_1 &= a\left(\hat{\boldsymbol{x}}\cos\phi_1+\hat{\boldsymbol{y}}\sin\phi_1\right)+\frac{d}{2}\hat{\boldsymbol{z}} \\
r_2 &= b\left(\hat{\boldsymbol{x}}\cos\phi_2+\hat{\boldsymbol{y}}\sin\phi_2\right)-\frac{d}{2}\hat{\boldsymbol{z}} \\
r^2=(r_2-r_1)^2 &= \left[(b\cos\phi_2-a\cos\phi_1)\hat{\boldsymbol{x}}+(b\sin\phi_2-a\sin\phi_1)\hat{\boldsymbol{y}}-d\hat{\boldsymbol{z}}\right]^2 \\
&= a^2+b^2+d^2-2ab\left(\cos\phi_1\cos\phi_2+\sin\phi_1\sin\phi_2\right) \\
&= a^2+b^2+d^2-2ab\cos(\phi_1-\phi_2) \\
\Rightarrow r &= \sqrt{a^2+b^2+d^2-2ab\cos(\phi_1-\phi_2)}
\end{aligned}$$

$$M_{12}=\frac{\mu}{4\pi}\oint_{\phi_2}d\phi_2\left(\oint_{\phi_1}d\phi_1\ \frac{ab\cos(\phi_1-\phi_2)}{\sqrt{a^2+b^2+d^2-2ab\cos(\phi_1-\phi_2)}}\right)$$

One of the integrals can be eliminated by performing the substitution

$u = \phi_1 - \phi_2$, $du = d\phi_1$

$$\begin{aligned} M_{12} &= \frac{\mu}{4\pi} \oint_{\phi_2} d\phi_2 \left(\oint_u du \, \frac{ab\cos(u)}{\sqrt{a^2 + b^2 + d^2 - 2ab\cos(u)}} \right) \\ &= \frac{\mu}{4\pi} \left(\oint_{\phi_2} d\phi_2 \right) \left(\oint_u du \, \frac{ab\cos(u)}{\sqrt{a^2 + b^2 + d^2 - 2ab\cos(u)}} \right) \\ &= \frac{\mu}{2} \oint_u du \, \frac{ab\cos(u)}{\sqrt{a^2 + b^2 + d^2 - 2ab\cos(u)}} \end{aligned}$$

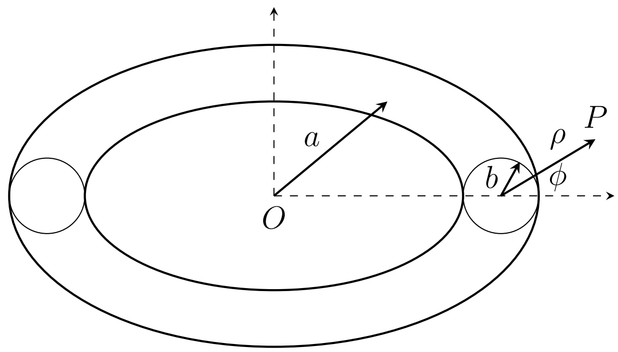
From a table:

$$\oint d\phi \frac{\cos\phi}{\sqrt{A - B\cos\phi}} = \frac{4\sqrt{A+B}}{B} \left[\left(1 - \frac{k^2}{2}\right) K(k) - E(k) \right], \quad k = \sqrt{\frac{2B}{A+B}}$$

Using the following substitutions:

$$\begin{aligned} A &\to \frac{a^2 + b^2 + d^2}{a^2 b^2} \\ B &\to \frac{2}{ab} \\ k &= \sqrt{\frac{4ab}{a^2 + b^2 + d^2 + 4}} \\ M_{12} &= \mu\sqrt{a^2 + b^2 + d^2 + 4} \left[\left(1 - \frac{k^2}{2}\right) K(k) - E(k) \right] \\ &= \mu \frac{2\sqrt{ab}}{k} \left[\left(1 - \frac{k^2}{2}\right) K(k) - E(k) \right] \\ &= \mu\sqrt{ab} \left[\left(\frac{2}{k} - k\right) K(k) - E(k) \right] \end{aligned}$$

3. A circular loop of mean radius a is made of wire having a circular cross section of radius b, with $b \ll a$. The sketch shows the relevant dimensions and coordinates for this problem.



(a) Using (5.37), the expression for the vector potential of a filamentary circular loop, and appropriate approximations for the elliptic integrals, show that the vector potential at the point P near the wire is approximately

$$A_\phi = (\mu_0 I/2\pi)[\ln(8a/\rho) - 2]$$

where ρ is the transverse coordinate shown in the figure and corrections are of order $(\rho/a)\cos\phi$ and $(\rho/a)^2$

Solution.

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\frac{4Ia}{\sqrt{a^2+r^2+2ar\sin\theta}}\left[\frac{(2-k^2)K(k)-2E(k)}{k^2}\right]$$

where k is defined as

$$k^2 = \frac{4ar\sin\theta}{a^2+r^2+2ar\sin\theta}$$

$$r = a + \rho\sin\phi$$
$$\Rightarrow a^2+r^2+2ar\sin\theta = a^2+(a+\rho\sin\phi)^2+2a(a+\rho\sin\phi)\sin\theta$$

For P close to the wire, $\sin\theta \approx 1$

$$\begin{aligned}\Rightarrow a^2+r^2+2ar\sin\theta &= a^2+(a+\rho\sin\phi)^2+2a(a+\rho\sin\phi)\\ &= 4a^2+\rho^2\sin^2\phi+4a\rho\sin\phi\end{aligned}$$

$$\begin{aligned}
k^2 &= \frac{4a^2 + a\rho\sin\phi}{4a^2 + \rho^2\sin^2\phi + 4a\rho\sin\phi} \\
&= \frac{4 + \frac{\rho}{a}\sin\phi}{4 + \frac{\rho^2}{a^2}\sin^2\phi + 4\frac{\rho}{a}\sin\phi} \\
&= 1 - 3\frac{\rho}{4a} + \frac{\rho^2}{2a^2} + \mathcal{O}\left(\frac{\rho^3}{a^3}\right)
\end{aligned}$$

$$\begin{aligned}
A_\phi(r,\theta) &= \frac{\mu_0}{4\pi}\frac{4Ia}{\sqrt{4a^2 + \rho^2\sin^2\phi + 4a\rho\sin\phi}}\left[\frac{(2-k^2)K(k) - 2E(k)}{k^2}\right] \\
&= \frac{\mu_0}{4\pi}\frac{4I}{\sqrt{4 + \frac{\rho^2}{a^2}\sin^2\phi + 4\frac{\rho}{a}\sin\phi}}\left[\frac{(1+\frac{3\rho}{4a})K(k) - 2E(k)}{1 - \frac{3\rho}{4a}}\right]
\end{aligned}$$

$$\begin{aligned}
K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} &= \frac{\pi}{2}\left\{1 + \frac{1}{4}\left(1 - \frac{3\rho}{4a}\right)\right\} \\
&= \frac{\pi}{8}\left(5 - \frac{3\rho}{4a}\right) \\
E(k) = \int_0^{\frac{\pi}{2}} d\theta\,\sqrt{1 - k^2\sin^2\theta} &= \frac{\pi}{2}\left\{1 - \frac{1}{4}\left(1 - \frac{3\rho}{4a}\right)\right\} \\
&= \frac{3\pi}{8}\left(1 + \frac{\rho}{4a}\right)
\end{aligned}$$

$$\begin{aligned}
A_\phi(r,\theta) &= \frac{\mu_0}{4\pi}\frac{4I}{\sqrt{4 + \frac{\rho^2}{a^2}\sin^2\phi + 4\frac{\rho}{a}\sin\phi}}\left[\frac{\pi}{8}\frac{(1+\frac{3\rho}{4a})(5 - 3\rho/4a) - 2(3 + 3\rho/4a)}{1 - \frac{3\rho}{4a}}\right] \\
&= \frac{\mu_0 I}{8}\left(\frac{1}{2} - \frac{1}{4}\frac{\rho}{a}\sin\phi\right)\left[\frac{(1+\frac{3\rho}{4a})(5 - 3\rho/4a) - 2(3 + 3\rho/4a)}{1 - \frac{3\rho}{4a}}\right] \\
&= \frac{\mu_0 I}{8}\left(\frac{1}{2} - \frac{1}{4}\frac{\rho}{a}\sin\phi\right)\left[\frac{-1 + 21\rho/4a}{1 - \frac{3\rho}{4a}}\right] \\
&= \frac{\mu_0 I}{8}\left(\frac{1}{2} - \frac{1}{4}\frac{\rho}{a}\sin\phi\right)\left[\frac{-1 + 21\rho/4a}{1 - \frac{3\rho}{4a}}\right] \\
&= \cdots
\end{aligned}$$

(b) Since the vector potential of part a is, apart from a constant, just that outside a straight circular wire carrying a current I, determine the vector potential inside the wire ($\rho < b$) in the same approximation by requiring continuity of A_ϕ and its radial derivative at $\rho = b$, assuming that the current is uniform in density inside the wire:

$$A_\phi = (\mu_0 I/4\pi)(1 - \rho^2/b^2) + (\mu_0 I/2\pi)[\ln(8a/b) - 2], \quad \rho < b$$

(c) Use (5.149) to find the magnetic energy, hence the self-inductance,

$$L = \mu_0 a[\ln(8a/b) - 7/4]$$

Are the corrections of order b/a or $(b/a)^2$? What is the change in L if the current is assumed to flow only on the surface of the wire (as occurs at high frequencies when the skin depth is small compared to b)?